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Shifts on a Deformed Hilbert Space

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In this paper we observe that the creation operator on a deformed Hilbert space is the product of an ordinary shift and a diagonal operator.

1. INTRODUCTION

We consider a Hilbert space which is spanned by the vectors $\{e_n, n = 0, 1, 2, \ldots\}$. These vectors are generated by the action of the *creation operator* a^* on the vector e_0 which we call a *vacuum* vector in the Hilbert space. The hermitian conjugate of a^* is the *annihilation operator* a. Together they satisfy the following relations:

$$\begin{aligned} aa^* &= qa^*a + 1 \\ \langle e_0, e_0 \rangle &= 1 \\ e_n &= (a^*)^n e_0 \\ ae_0 &= 0 \end{aligned} \tag{1}$$

We call this Hilbert space a *deformed Hilbert space* and denote it by H_q, where q is a deforming parameter ranging over $0 < q < 1$. This space was discussed in ref. 1.

Using the relations in (1), we observe the following:

$$\begin{aligned} a^*e_n &= e_{n+1} \\ ae_n &= [n]e_{n-1} \end{aligned} \tag{2}$$

where we take $[n] = (1 - q^n)/(1 - q) = 1 + q + q^2 + \ldots + q^{n-1}$.

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To prove $\langle e_m, e_n \rangle = [n]!\delta_{nm}$ we proceed to prove $\langle e_n, e_n \rangle = [n]!$:

$$\begin{aligned} \|e_{n+1}\|^2 &= \langle a^* e_n, e_{n+1} \rangle \\ &= \langle e_n, a e_{n+1} \rangle \\ &= \langle e_n, [n+1] e_n \rangle \\ &= [n+1] \|e_n\|^2 \end{aligned}$$

Hence, $\|e_n\|^2 = [n]\|e_{n-1}\|^2 = [n][n-1] \ldots [2][1]\|e_0\|^2 = [n]!\|e_0\|^2$.

Since by (1), $\|e_0\|^2 = 1$, we have

$$\langle e_n, e_n \rangle = \|e_n\|^2 = [n]!$$

Now,

$$\begin{aligned} \langle e_n, e_m \rangle &= \langle a^{*n} e_0, a^{*m} e_0 \rangle = \langle e_0, a^n a^{*m} e_0 \rangle \\ &= 0 \qquad \text{if} \quad m > n \quad \text{or} \quad m < n \end{aligned}$$

Thus,

$$\langle e_n, e_m \rangle = [n]!\, \delta_{nm}$$

Hence, the following can be derived using relations (1):

$$\begin{aligned} a^* e_n &= e_{n+1} \\ a e_n &= [n] e_{n-1} \\ \langle e_n, e_m \rangle &= [n]!\, \delta_{nm} \end{aligned} \tag{3}$$

where $[n]! = [n] \ldots [2][1]$; $[0]! = 1$.

The vectors $\{([n]!)^{-1/2} e_n\}$ form an orthonormal basis and the Hilbert space H_q consists of all vectors $f = \sum_{n=0}^{\infty} f_n e_n$ with complex f_n such that

$$\langle f, f \rangle = \sum_{n=0}^{\infty} |f_n|^2 [n]!$$

is finite.

If $g = \sum_{n=0}^{\infty} g_n e_n$ is also a vector in the space, then

$$\langle f, g \rangle = \sum_{n=0}^{\infty} \bar{f}_n g_n [n]!$$

where the bar denotes the complex conjugate.

In this paper we prove two theorems. The content of the first theorem is that the shift operator is always a bounded operator in H_q. The content of the second theorem is that the shift operator on H_q is unitarily equivalent to the weighted shift on l^2. Using these theorems, we have shown that the creation operator mentioned above is the product of an ordinary shift and a diagonal operator [2].

2. THEOREMS

Theorem 1. The shift is always an operator in H_q, that is, if $f = \Sigma_{n=0}^{\infty} f_n e_n = \{f_0, f_1, f_2, \ldots\} \in H_q$, then $Sf = \{0, f_0, f_1, \ldots\} \in H_q$ and as f varies over H_q, $\|Sf\|$ is bounded by a constant multiple of $\|f\|$.

Proof. It is necessary that $\|e_{n+1}\| \le \alpha\|e_n\|$, where e_n is the vector whose coordinate with index n is 1 and all other coordinates are 0. Since

$$\|e_n\|^2 = \langle e_n, e_n\rangle = [n]! \equiv p_n$$

this condition says that

$$\frac{p_{n+1}}{p_n} = \frac{[n+1]!}{[n]!} = [n+1] = \frac{1-q^{n+1}}{1-q} \to \frac{1}{1-q} \equiv \alpha$$

as $n \to \infty$.

Thus the condition $\|e_{n+1}\| \le \alpha\|e_n\|$ says that the sequence $\{p_{n+1}/p_n\}$ is bounded with bound $1/(1-q)$.

This condition is also sufficient. For, if $p_{n+1}/p_n \le 1/(1-q)$ for all n, then

$$\begin{aligned} \|Sf\|^2 &= \sum_{n=1}^{\infty} p_n |f_{n-1}|^2 \\ &= \sum_{n=1}^{\infty} \frac{p_n}{p_{n-1}} p_{n-1} |f_{n-1}|^2 \\ &\le \frac{1}{1-q} \sum_{n=0}^{\infty} p_n |f_n|^2 \\ &= \frac{1}{1-q} \|f\|^2 \end{aligned}$$

Hence

$$\|Sf\| \le \frac{1}{(1-q)^{1/2}} \|f\|$$

Theorem 2. The shift S on H_q is unitarily equivalent to the weighted shift T, with weights $\{\sqrt{[n+1]}\}$, on l^2.

Proof. We observe that $\{[n]!\}$ is a sequence of positive numbers with $\{[n+1]!/[n]!\}$ a bounded sequence.

If $f = \{f_0, f_1, f_2, \ldots\} \in H_q$, we write
$Uf = \{1.f_0, \sqrt{[1]!}f_1, \sqrt{[2]!}f_2, \ldots\}$. Then,

$$\|Uf\|^2 = \|f\|^2 = \sum_{n=0}^{\infty} [n]![f_n]^2 < \infty$$

Thus, $Uf \in l^2$. Hence, U maps H_q into l^2 linearly.

If $g = \{g_0, g_1, g_2, \ldots\} \in l^2$ and if $f_n = g_n/\sqrt{[n]!}$, then $\Sigma_{n=0}^{\infty} [n]!|f_n|^2 = \Sigma_{n=0}^{\infty} |g_n|^2 < \infty$. This proves that U maps H_q onto l^2.

Now,

$$\begin{aligned} USU^{-1}\{g_0, g_1, g_2, \ldots\} &= US\left\{\frac{g_0}{\sqrt{[0]!}}, \frac{g_1}{\sqrt{[1]!}}, \frac{g_2}{\sqrt{[2]!}}, \ldots\right\} \\ &= U\left\{0, \frac{g_0}{\sqrt{[0]!}}, \frac{g_1}{\sqrt{[1]!}}, \frac{g_2}{\sqrt{[2]!}}, \ldots\right\} \\ &= \left\{0, \sqrt{\frac{[1]!}{[0]!}}\, g_0, \sqrt{\frac{[2]!}{[1]!}}\, g_1, \sqrt{\frac{[3]!}{[2]!}}\, g_2, \ldots\right\} \\ &= T\{g_0, g_1, g_2, \ldots\} \end{aligned}$$

Thus, U transforms S onto T. That is, the transform of the ordinary shift on H_q is a weighted shift on l^2.

Observation. Because of Theorem 2, a shift S on H_q is the weighted shift T, with weights $\{\sqrt{[n+1]}\}$ on l^2; we have $T = UD$, where U is an ordinary shift and D is a diagonal operator with diagonals $\{[n+1]!/[n]!\}$. We see that

$$a^* = UD$$

and

$$ae_n = D^*U^*e_n = D^*e_{n-1} = [n]e_{n-1}$$

We also see that

$$\|a\| = \|a^*\| = sup_n\sqrt{[n+1]} = sup_n\sqrt{\frac{1-q^{n+1}}{1-q}} = (1-q)^{-1/2}$$

REFERENCES

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